Auditorium Exercise Sheet 3

Differential Equations I for Students of Engineering Sciences

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Review of differential equations

A (real, scalar) ODE is an equation in which a function $y = y(t)$ and its derivatives $y', y'', \dots, y^{(m)}$ (up to order $m \in \mathbb{N}$) are related:

$$F(t, y, y', y'', \dots, y^{(m)}) = 0 \to m\text{-th order ODE in } \textbf{implicit form}$$

$$y^{(m)} = f(t, y, y', y'', \dots, y^{(m-1)}) \to m\text{-th order ODE in } \textbf{explicit form}$$

where $y : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ domain of definition.

Example 1

- The differential equation $6t^2y'' + \ln(t+8)(y')^2 - 5 = 0$ for $t > 0$ is given in implicit form. The corresponding explicit expression is $y'' = (5 - \ln(t+8)(y')^2)/6t^2$, well-defined for all $t > 0$.
- $y^{(4)} = 2t\sin(t^3)/5 - y'e^t/2 - 6y''/10$ for $t \in \mathbb{R}$ is in explicit form. The corresponding implicit form is $10y^{(4)} - 4t\sin(t^3) + 5y'e^t + 6y'' = 0$.

Resolution by separation of variables

A first order ODE of the form

$$y'(t) = g(t)h(y(t))$$

with g, h continuous real functions is called a separable variables differential equation.

In case $h(y(t)) \neq 0$, we can solve it dividing both sides by $h(y)$ and then integrating with respect to the independent variable t:

$$\int \frac{\mathrm{d}y}{h(y)} \underset{y \mapsto y(t)}{=} \int \frac{y'(t)}{h(y(t))}\mathrm{d}t = \int g(t)\mathrm{d}t$$

After integrating, solve with respect to y.

Example 2

Solve the ODE $y'(t) = 2ty^3(t)$ under initial condition $y(0) = 1$.
Which is the largest interval in which the solution is defined?

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Solve the ODE $\quad y'(t) = 2ty^3(t) \quad$ under initial condition $y(0) = 1$.
Which is the largest interval in which the solution is defined?

It is a separable variable ODE with $g(t) := 2t$ and $h(y) := y^3$. Notice that $y \equiv 0$ is a solution of the equation, but it does NOT satisfy the initial condition. Suppose then $y \neq 0$ and compute:

$$\int \frac{\mathrm{d}y}{y^3} \underset{y \mapsto y(t)}{=} \int \frac{y'(t)}{y^3(t)} \mathrm{d}t = \int 2t \mathrm{d}t$$

$$-\frac{1}{2y^2} = t^2 + C_1 \implies y(t) = \pm \frac{1}{\sqrt{C_2 - 2t^2}} \rightarrow \text{gen. sol. of the ODE}$$

Employ now the initial condition:

$$1 = y(0) = \pm \frac{1}{\sqrt{C_2 - 2 \cdot 0^2}} \implies C_2 = 1 \text{ and } y(t) = \frac{1}{\sqrt{1 - 2t^2}}.$$

Example 2

Solve the ODE $\qquad y'(t) = 2ty^3(t) \qquad$ under initial condition $y(0) = 1$.
Which is the largest interval in which the solution is defined?

The solution $y(t) = \frac{1}{\sqrt{1-2t^2}}$ is well-defined where $1 - 2t^2 > 0$, namely in the interval $I = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

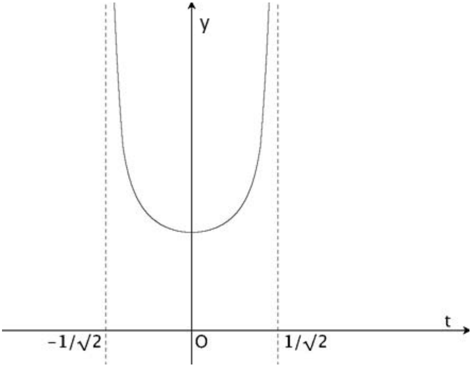

Figure: Plot of $y(t) = \frac{1}{\sqrt{1-2t^2}}$

Resolution by substitution

Similarity equation

A first order ODE of the type

$$y' = \psi\left(\frac{at + by + c}{dt + ey + f}\right)$$

with $a, \dots, f \in \mathbb{R}$ is called a **similarity equation** (or Jacobi equation).

In particular, the following cases are easy to solve by applying a change of variables:

- If $y' = \psi(at + by + c)$, set $\boldsymbol{u(t) := at + by + c}$ to get $u' = b\psi(u) + a$;
- If $y' = \psi\left(\frac{y}{t}\right)$, set $\boldsymbol{u(t) := \frac{y}{t}}$ to get $u' = \frac{\psi(u) - u}{t}$.

In both cases we obtain an ODE with separable variables - to be solved in u!

Example 3

Find the general solution of the ODE

$$y' = \frac{t^4 + y^4}{ty^3}, \qquad t \neq 0.$$

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Rewriting the equation it is $y' = \left(\frac{t}{y}\right)^3 + \frac{y}{t}$, i.e. a similarity equation with $\psi(u) = u + u^{-3}$ (with $a = c = e = f = 0$, and $b = d = 1$).
We substitute $u(t) := \frac{y}{t} \implies y(t) = tu(t) \implies y'(t) = u(t) + tu'(t)$.

We obtain the separable variable ODE: $\cancel{u} + tu' = \cancel{u} + u^{-3}$

$$\int u^3 \mathrm{d}u = \int u^3 u' \mathrm{d}t = \int \frac{1}{t} \mathrm{d}t$$

$$u^4(t) = 4\ln|t| + C \implies u(t) = \pm(4\ln|t| + C)^{1/4}.$$

Finally, substitute back: $y(t) = tu(t) = \pm t(4\ln|t| + C)^{1/4}$.

From ODEs to systems

Given a (scalar) ODE of any order $m \in \mathbb{N}$ (for example, in explicit form)

$$y^{(m)} = f(t, y, y', y'', \dots, y^{(m-1)}) \qquad \text{for } y : I \to \mathbb{R}, \tag{1}$$

we introduce the functions $y_1, y_2, \dots, y_m : I \to \mathbb{R}$ defined as

$$\mathbf{y_1 := y, y_2 := y_1{}' = y', y_3 := y_2{}' = y'', \dots, y_m := (y_{m-1})' = y^{(m\text{-}1)}},$$

thus $y_m{}' = y^{(m)}$. We may rewrite the ODE in (1) as

$$y_m{}' = f(t, y_1, y_2, y_3, \dots, y_m)$$

If we consider the conditions given by the definitions of $y_1, \dots, y_m$, we obtained a system of m ODEs of order 1:

$$\begin{cases} y_1{}' = y_2 \\ y_2{}' = y_3 \\ \vdots \qquad \vdots \\ (y_{m-1})' = y_m \\ (y_m)' = f(t, y_1, y_2, y_3, \dots, y_m) \end{cases}$$

Linear ODEs of m-th order as linear systems

In case the ODE in (1) is linear of order m and in the explicit form, i.e. it can be written as as

$$y^{(m)} = b(t) - a_0(t)y - a_1(t)y' - \cdots - a_{m-1}(t)y^{(m-1)} \tag{2}$$

with $a_0, a_1, \ldots, a_{m-1}, b : I \to \mathbb{R}$ functions on $I \subseteq \mathbb{R}$, then y is solution of (2) if and only if $(y_1, y_2, \ldots, y_m)$ solves the system

$$\begin{cases} {y_1}' = y_2 \\ {y_2}' = y_3 \\ \vdots \qquad \vdots \\ (y_{m-1})' = y_m \\ {y_m}' = b(t) - a_0(t)y - a_1(t)y' - \cdots - a_{m-1}(t)y^{(m-1)} \end{cases} \tag{3}$$

Let $\mathbf{Y}(t) := \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{pmatrix}$ and $\mathbf{B}(t) := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b(t) \end{pmatrix}$ be vectors of n components,

$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \\ -a_0(t) & -a_1(t) & \dots & -a_{m-1}(t) \end{pmatrix}$ matrix of order n.

Rewrite (3) as

$$\mathbf{Y}'(t) = \mathbf{A}(t) \cdot \mathbf{Y}(t) + \mathbf{B}(t).$$

- The following bijection holds:

$$\left\{ \begin{matrix} \text{linear } m\text{-th order ODEs} \\ \text{in explicit form} \end{matrix} \right\} \Longleftrightarrow \left\{ \begin{matrix} \text{linear systems of } m \text{ ODEs} \\ \text{of order 1} \end{matrix} \right\}$$

$$y^{(m)} = b(t) - \sum_{i=0}^{m-1} a_i(t) y^{(i)}(t) \leftrightarrow \quad \mathbf{Y}'(t) = \mathbf{A}(t) \cdot \mathbf{Y}(t) + \mathbf{B}(t).$$

Example 4

Rewrite the following IVP of a third order ODE

$$\begin{cases} 3y''' + 4t\cos(2t)y' - e^t y + 6t = 12, \ \ t > 5; \\ y(5) = -1, \ \ y'(5) = 0, \ \ y''(5) = 2 \end{cases} \tag{4}$$

as an initial value problem for a first order system.

Example 4

Rewrite the following IVP of a third order ODE

$$\begin{cases} 3y''' + 4t\cos(2t)y' - e^t y + 6t = 12, \ \ t > 5; \\ y(5) = -1, \ \ y'(5) = 0, \ \ y''(5) = 2 \end{cases} \tag{4}$$

as an initial value problem for a first order system.
Set $y_1 := y, \ y_2 := {y_1}' = y', \ y_3 := {y_2}' = y'' \rightarrow {y_3}' = y'''$.

Substituting into (4) returns : $3{y_3}' + 4t\cos(2t)y_2 - e^t y_1 + 6t = 12$
$\implies {y_3}' = (-4t\cos(2t)y_2 + e^t y_1 + 12 - 6t)/3.$

$$\begin{pmatrix} {y_1}' \\ {y_2}' \\ {y_3}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ e^t/3 & -4t\cos(2t)/3 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 4 - 2t \end{pmatrix}$$

$$\mathbf{Y'} = \mathbf{A} \quad \mathbf{Y} + \mathbf{B}$$

with $\mathbf{Y}(5) = \begin{pmatrix} y_1(5) \\ y_2(5) \\ y_3(5) \end{pmatrix} = \begin{pmatrix} y(5) \\ y'(5) \\ y''(5) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

Exercise 1

Consider the differential equation

$$\sin(6t)y' - t^4 y''' - 3ty'' + 3y/4 = 2e^{2t}. \tag{5}$$

(i) Determine the order of the ODE and if it is linear/non-linear, homogeneous/inhomogeneous.

(ii) Write (5) in explicit form in the domain $\{t \in \mathbb{R} : t > 0\}$.

(iii) Given the initial values $y^{(i)}(0) = z_i$ for $i \in \{0, 1, 2\}$ and $z_i \in \mathbb{R}$, rewrite (5) as a system of first order ODEs with conditions.

Exercise 2

Find the general solution of each of the following separable variables ODEs, then determine the respective solutions of the related problem under the constraint $y(1) = 1/2$.

(i) $y' = 6t^2 y$;

(ii) $y' = 4t^3\sqrt{t},\ t > 0$;

(iii) $y' + \dfrac{2x}{y}(1 + 2x^2) = 0$;

(iv) $y' = y^2 - 1$;

(v) $ty' = \sqrt{1 - y^2},\ t \in (1, 2)$.

Exercise 3

Determine the right substitution which transforms each of the following similarity ODEs into a separable variables equation, then solve the equation obtained.

$$(i)\ \dot{x} = \frac{t^2 + x^2}{tx},\ \ t \neq 0;$$

$$(ii)\ y' = \frac{y + 2x}{x},\ \ x \neq 0;$$

$$(iii)\ y' = (1 + 4x + 16y)^2.$$

Appendix

Table of most common integrals

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$
2. $\int \frac{1}{x}\,dx = \ln|x|$
3. $\int e^x\,dx = e^x$
4. $\int a^x\,dx = \frac{a^x}{\ln a}$
5. $\int \sin x\,dx = -\cos x$
6. $\int \cos x\,dx = \sin x$
7. $\int \sec^2 x\,dx = \tan x$
8. $\int \csc^2 x\,dx = -\cot x$
9. $\int \sec x \tan x\,dx = \sec x$
10. $\int \csc x \cot x\,dx = -\csc x$
11. $\int \sec x\,dx = \ln|\sec x + \tan x|$
12. $\int \csc x\,dx = \ln|\csc x - \cot x|$
13. $\int \tan x\,dx = \ln|\sec x|$
14. $\int \cot x\,dx = \ln|\sin x|$
15. $\int \sinh x\,dx = \cosh x$
16. $\int \cosh x\,dx = \sinh x$
17. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

*19. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

*20. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

AUDITORIUM EXERCISE CLASS 3

Exercise 3

(i) $x = x(t)$, $\dot{x}(t) = \dfrac{t^2 + x^2}{tx}$, $t \neq 0$

$u(t) := \dfrac{x}{t} \to x = t \cdot u$

$\left[\dot{x} = \dfrac{t^2}{tx} + \dfrac{x^2}{tx} = \dfrac{t}{x} + \dfrac{x}{t}\right] \rightsquigarrow$

$\dot{x}(t) = \dfrac{dx}{dt} = \dfrac{d(t \cdot u)}{dt} = u(t) + t u'(t)$

$\dot{x} = u + t \cdot u'$, $\quad \dfrac{t}{x} + \dfrac{x}{t} = \dfrac{1}{u} + u$

$\Downarrow$

separable variables ODE in u: $\left[t \cdot u'(t) = \dfrac{1}{u}\right] \rightsquigarrow u'(t) = \left(\dfrac{1}{t}\right)\left(\dfrac{1}{u}\right)$, with $g(t) = \dfrac{1}{t}$, $h(u) = \dfrac{1}{u}$

Solve the new ODE:

$$c_1 + \frac{u^2}{2} = \int u\, du = \int u \cdot \underbrace{u' dt}_{du} = \int \frac{1}{t} dt = \ln|t| + c_2$$

$$\frac{u^2}{2} = \ln|t| + \tilde{c} \Rightarrow u(t) = \pm\sqrt{2\ln|t| + 2\tilde{c}} \Rightarrow \left[u(t) = \pm\sqrt{2\ln|t| + C}\right], \quad C \in \mathbb{R}$$

$\tilde{c} := c_2 - c_1$, $\quad C = 2\tilde{c}$

$$\Rightarrow \left[x(t) = t \cdot u(t) = \pm t\sqrt{2\ln|t| + C}\right]$$

$\rightsquigarrow$ If possible, check that it's a solution!

$\dot{x}(t) = \dots$

(ii) $\left[y' = \dfrac{y + 2x}{x}\right]$ for $y = y(x)$, $x \neq 0$

$y' = \dfrac{y}{x} + 2$

$u(x) := \dfrac{y}{x} \to y(x) = x \cdot u(x)$

$y' = u + x \cdot u'$

$u + x \cdot u' = u + 2$

$\left[u'(x) = \dfrac{2}{x}\right]$ separable var. ODE in u

$$\int du = \int u'(x)\, dx = \int \frac{2}{x} dx \Rightarrow \left[u = 2\ln|x| + c\right] \overset{y = x \cdot u}{\Rightarrow} \left[y(x) = x \cdot (2\ln|x| + c), \quad c \in \mathbb{R}\right]$$

Exercise 1

$$\sin(6t)\cdot y - t^4 y''' - 3t\cdot y'' + \frac{3}{4}y = 2e^{2t} \quad \text{in } I := \{t>0\}$$

(i) ODE of order $m=3$, linear, inhomogeneous
$\hookrightarrow b(t) = 2e^{2t}$

(ii) Explicit form = ?
$\hookrightarrow y''' = \dots$

Divide by the coeff. of y''' ($-t^4 \neq 0$):

$$y''' = -\frac{3t}{t^3}y'' + \frac{\sin(6t)}{t^4}y' + \frac{3}{4t^4}y - \frac{2e^{2t}}{t^4} \quad \text{in } \{t>0\}$$

(iii) Rewriting as system:

$$(Y_3)' = \frac{-3}{t^3}\cdot Y_3 + \frac{\sin(6t)}{t^4}\cdot Y_2 + \frac{3}{4t^4}\cdot Y_1 - \frac{2e^{2t}}{t^4}$$

Initial conditions
$$\begin{cases} y^{(0)}(0) = y(0) = z_0 \\ y^{(1)}(t) = y'(0) = z_1 \\ y^{(2)}(t) = y''(0) = z_2 \end{cases}$$

Set $Y_1 := y$
$Y_2 := Y_1' = y'$
$Y_3 := Y_2' = y''$ ← $y''' = Y_3'$

$$Y := \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \rightsquigarrow Y' = \begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{3}{4t^4} & \frac{\sin(6t)}{t^4} & -\frac{3}{t^3} \end{pmatrix}}_{A} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ -\frac{2e^{2t}}{t^4} \end{pmatrix}}_{B}$$

← ODE in matricial form

$$Y' = A\cdot Y + B$$

$Y_1(0) = z_0$
$Y_2(0) = z_1$
$Y_3(0) = z_2$

$$\rightarrow Y(0) = \begin{pmatrix} Y_1(0) \\ Y_2(0) \\ Y_3(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$$

← initial conditions in matricial form